MATH 2050 - Subsequences & Bolzano-Weierstrass Thm

(Reference: Bartle § 3.4)

<u>Def</u>

Def$^{\underline{n}}$: Let $(x_n)_{n\in\mathbb{N}}$ be a seq. of real numbers.

Suppose $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing seq. of natural no..

THEN,

$$(x_{n_k})_{k\in\mathbb{N}} := (x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots)$$

is called a subsequence of $(x_n)_{n\in\mathbb{N}}$.

k^{th} term of (x_{n_k}) $=$ n_k^{th} term of (x_n)

Intuitively:

$$(x_n) = (x_1, x_2, x_3, x_4, x_5, x_6, \ldots)$$

$$(x_{n_k}) = (x_1, x_2, x_4, x_6, \ldots\ldots)$$

$k=1$	$k=2$	$k=3$	$k=4$
$n_1=1$	$n_2=2$	$n_3=4$	$n_4=6$

E.g.) (Tail of a seq.) For each fixed $\ell \in \mathbb{N}$, then the ℓ-tail $(x_{k+\ell})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$

(Here, $n_k = k + \ell$)

E.g.) $(x_n) = ((-1)^n)$

Then $(1, 1, 1, \ldots, 1, \ldots)$ is a subseq.

Thm: Suppose $\lim_{n\to\infty} x_n = x$. Then, every subseq. (x_{n_k}) of (x_n) also converges to the same limit, i.e. $\lim_{k\to\infty} x_{n_k} = x$.

Proof: Note: $n_k \geq k$ for all $k \in \mathbb{N}$ (by induction).

Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim_{n\to\infty} x_n = x \implies \exists K \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \quad \forall n \geq K$

By Note above, when $k \geq K$, then $n_k \geq k \geq K$. Thus,

$$|x_{n_k} - x| < \varepsilon \qquad \forall k \geq K$$

$\square$

Example: Show that $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$ for $c > 1$.

Pf: Let $z_n := c^{\frac{1}{n}}$. Then, by induction,

(z_n) is decreasing and bdd below by 1

By MCT, $\lim_{n\to\infty} (z_n) =: z$ exists.

Consider the subseq. $(z_{n_k})_{k\in\mathbb{N}} := (z_{2k})$, by Thm above,

$$\lim_{k\to\infty} (z_{n_k}) = z.$$

$\because z_n > 1 \ \forall n \in \mathbb{N}$

rejected

Now, $z_{2n} = c^{\frac{1}{2n}} = (c^{\frac{1}{n}})^{\frac{1}{2}} = (z_n)^{\frac{1}{2}}$

Take $n\to\infty$ on both sides, we have $z = \sqrt{z} \implies z = \cancel{0}$ or 1. $\square$

In summary.

MCT: (x_n) monotone + bdd $\Rightarrow$ (x_n) convergent.

Thm: (x_n) convergent $\underset{\not\Leftarrow}{\Rightarrow}$ (x_n) bdd

Thm: (x_n) convergent $\underset{\Leftarrow}{\Rightarrow}$ ANY subseq. (x_{n_k}) of (x_n) converge to the SAME limit.

Take negation yields two divergence criteria:

Cor: (x_n) unbdd $\Rightarrow$ (x_n) divergent

Cor: Either: $\exists$ subseq (x_{n_k}) which is divergent

or : $\exists$ two subseq (x_{n_k}) and $(x_{n_{k'}})$ s.t

$$\lim_{k\to\infty} (x_{n_k}) \neq \lim_{k\to\infty} (x_{n_{k'}})$$

$\Longrightarrow$ (x_n) divergent.

Example: $((-1)^n)$ is divergent since $\exists$ two subseq.

$$(1, 1, 1, 1, \dots, 1) \to 1$$

$$(-1, -1, -1, -1, \dots, -1) \to -1$$

Example: $\left(\cos \frac{n\pi}{2}\right) = (0, -1, 0, 1, 0, -1, 0, 1, \dots)$

$\exists$ subseq. $(0, 0, \dots, 0) \to 0$.

$(-1, 1, -1, 1, \dots)$ divergent $\Rightarrow$ original seq is divergent.

Example: $(x_n) = (0, 1, 0, 2, 0, 3, 0, \dots, 0, n, \dots)$ divergent

since $\exists$ subseq. $(1, 2, 3, 4, \dots, n, \dots)$ unbdd $\Rightarrow$ divergent.

Recall: (x_n) divergent $\Leftrightarrow$ (x_n) DOES NOT converge to x for ANY $x \in \mathbb{R}$.

Thm: Fix $x \in \mathbb{R}$. Then

(x_n) does NOT converge to x — either (x_n) divergent or $(x_n) \to x' \neq x$

$\Leftrightarrow$ $\exists\, \varepsilon_0 > 0$ AND a subseq. (x_{n_k}) of (x_n) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Proof: Recall:

$$\lim_{n\to\infty} (x_n) = x \iff \forall \varepsilon > 0,\ \exists K = K(\varepsilon) \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \quad \forall n \geq K$$

Negate the above.

(x_n) does NOT converge to x $\Leftrightarrow$ $\exists\, \varepsilon_0 > 0$ s.t. $\forall K \in \mathbb{N}$ s.t. $\exists\, n_K \geq K$ s.t. $|x_{n_K} - x| \geq \varepsilon_0$

- Take $K = 1$, choose $n_1 \geq 1$ s.t. $|x_{n_1} - x| \geq \varepsilon_0$
- Take $K = n_1 + 1$, choose $n_2 \geq n_1 + 1$ s.t. $|x_{n_2} - x| \geq \varepsilon_0$

repeat $\rightsquigarrow$ $(x_{n_k})_{k\in\mathbb{N}}$ s.t. $|x_{n_k} - x| \geq \varepsilon_0$

$\blacksquare$

Recall: "MCT": (x_n) bdd & monotone $\Rightarrow$ (x_n) convergent

[E.g.) $(x_n) = ((-1)^n)$ bdd, but NOT monotone, NOT convergent.]

Q: What if (x_n) is ONLY bdd ?

Bolzano-Weierstrass Thm: "BWT"

"compactness" (MATH 3070).

(x_n) bdd $\Rightarrow$ $\exists$ subseq. (x_{n_k}) which is convergent.

But not unique!

Example: $(x_n) = ((-1)^n)$ has a convergent subseq.

namely $(x_{2k}) = (1, 1, 1, 1, \dots) \to 1$

another choice $(x_{2k-1}) = (-1, -1, -1, -1, \dots) \not\to -1$

Proof: We will prove it using "Nested Interval Property" (NIP)

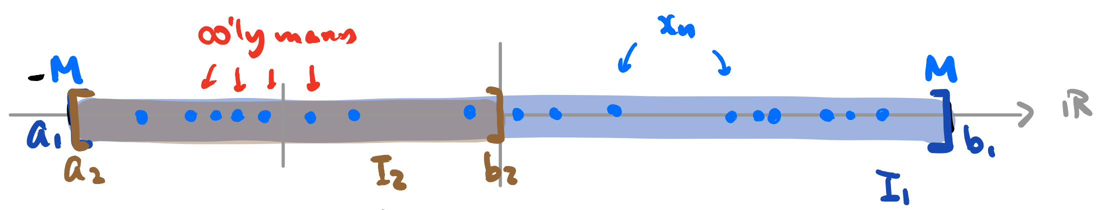
Recall: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ nested, closed & bdd

$\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$. If furthermore $\lim \text{Length}(I_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Goal: Construct I_n inductively satisfying the hypothesis above.

Given a bdd seq (x_n), by def^{n}, $\exists M > 0$ s.t. $|x_n| \leq M$ $\forall n \in \mathbb{N}$

i.e $\forall n \in \mathbb{N}$, $x_n \in [-M, M] =: I_1 = [a_1, b_1]$



Do "method of bisection":

Consider the midpoint $\frac{a_1 + b_1}{2}$, then

Case 1: $[a_1, \frac{a_1+b_1}{2}]$ contains infinitely many terms of (x_n)

$\rightsquigarrow$ choose $I_2 := [a_1, \frac{a_1+b_1}{2}] = [a_2, b_2]$.

Case 2: Otherwise $\rightsquigarrow$ choose $I_2 := [\frac{a_1+b_1}{2}, b_1] = [a_2, b_2]$

Repeat the process. take a mid pt. $\frac{a_2+b_2}{2}$, choose $I_3 = [a_3, b_3]$.

Inductively, we obtain a seq of intervals :

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots\cdots \quad \text{nested, closed \& bdd}$$

s.t. • each I_n contains infinitely many terms of (x_n)

• $\text{Length}(I_n) = \frac{2M}{2^{n-1}} \to 0$ as $n \to \infty$

By "NIP", $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$

Claim: $\exists$ subseq. $(x_{n_k}) \to \xi$

Pf: Take any $x_{n_1} \in I_1$, then since I_2 contains infinitely many terms of (x_n)

$\rightsquigarrow$ we can choose $n_2 > n_1$ s.t $x_{n_2} \in I_2$

$\rightsquigarrow$ keep on doing this, we obtain $n_1 < n_2 < n_3 < \cdots$ st

$$x_{n_k} \in I_k = [a_k, b_k] \qquad \forall k \in \mathbb{N}.$$

i.e. $$a_k \le x_{n_k} \le b_k. \qquad \forall k \in \mathbb{N}.$$

Now, $\bigcap_{n=1}^{\infty} I_n = \{\xi\} \Rightarrow \lim a_k = \lim b_k = \xi$.

By Squeeze Thm, we have $\lim_{k\to\infty} (x_{n_k}) = \xi$.

$\square$

As an application of BWT, we prove:

Prop: Let (x_n) be a bdd sequence.

$(x_n) \to x \iff$ ANY convergent subseq. (x_{n_k}) has $\lim_{k\to\infty} (x_{n_k}) = x$

Proof: "$\Rightarrow$" DONE.

"$\Leftarrow$" Suppose NOT, ie (x_n) does NOT converge to x.

By earlier thm, $\exists\ \varepsilon_0 > 0$ & a subseq. (x_{n_k}) s.t.

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall k \in \mathbb{N} \qquad (*)$$

By BWT., $(x_{n_k})_k$ bdd $\Rightarrow \exists$ convergent subseq. $(x_{n_{k_\ell}})_\ell$ of $(x_{n_k})_k$

$(\because (x_n)$ bdd$)$ which is also a subseq of $(x_n)_n$

By hypothesis, $\lim_{\ell\to\infty} (x_{n_{k_\ell}}) = x$ contradicting $(*)$. $\blacksquare$